Solution for HW 1 — 20 - 9 - 2016

§3) 8) The formula is clearly true for $n=1$.
Assume that it is true for $n=m$.
i.e. $(z_1+z_2)^m = \sum_{k=0}^{m} \binom{m}{k} z_1^{m-k} z_2^k$.
For $n=m+1$,

$$
\begin{aligned}
(z_1+z_2)^{m+1} &= (z_1+z_2)(z_1+z_2)^m \\
&= (z_1+z_2)\left(\sum_{k=0}^{m}\binom{m}{k} z_1^{m-k} z_2^k\right) \quad \text{(by assumption)} \\
&= \sum_{k=0}^{m}\binom{m}{k} z_1^{m+1-k} z_2^k + \sum_{k=0}^{m}\binom{m}{k} z_1^{m-k} z_2^{k+1} \\
&= z_1^{m+1} + \sum_{k=1}^{m}\binom{m}{k} z_1^{m+1-k} z_2^k + \sum_{k=0}^{m-1}\binom{m}{k} z_1^{m-k} z_2^{k+1} + z_2^{m+1} \\
&= z_1^{m+1} + \sum_{k=1}^{m}\binom{m}{k} z_1^{m+1-k} z_2^k + \sum_{k=1}^{m}\binom{m}{k-1} z_1^{m+1-k} z_2^k + z_2^{m+1} \\
&= z_1^{m+1} + \sum_{k=1}^{m}\left(\binom{m}{k}+\binom{m}{k-1}\right) z_1^{m+1-k} z_2^k + z_2^{m+1}
\end{aligned}
$$

Since
$$
\begin{aligned}
\binom{m}{k}+\binom{m}{k-1} &= \frac{m!}{k!(m-k)!} + \frac{m!}{(k-1)!(m-k+1)!} \\
&= \frac{m!(m-k+1+k)}{k!(m-k+1)!} \\
&= \frac{m!(m+1)}{k!(m-k+1)!} \\
&= \frac{(m+1)!}{k!(m-k+1)!} \\
&= \binom{m+1}{k},
\end{aligned}
$$

we have
$$
\begin{aligned}
(z_1+z_2)^{m+1} &= z_1^{m+1} + \sum_{k=1}^{m}\left(\binom{m}{k}+\binom{m}{k-1}\right) z_1^{m+1-k} z_2^k + z_2^{m+1} \\
&= z_1^{m+1} + \sum_{k=1}^{m}\binom{m+1}{k} z_1^{m+1-k} z_2^k + z_2^{m+1} \\
&= \sum_{k=0}^{m+1}\binom{m+1}{k} z_1^{m+1-k} z_2^k.
\end{aligned}
$$

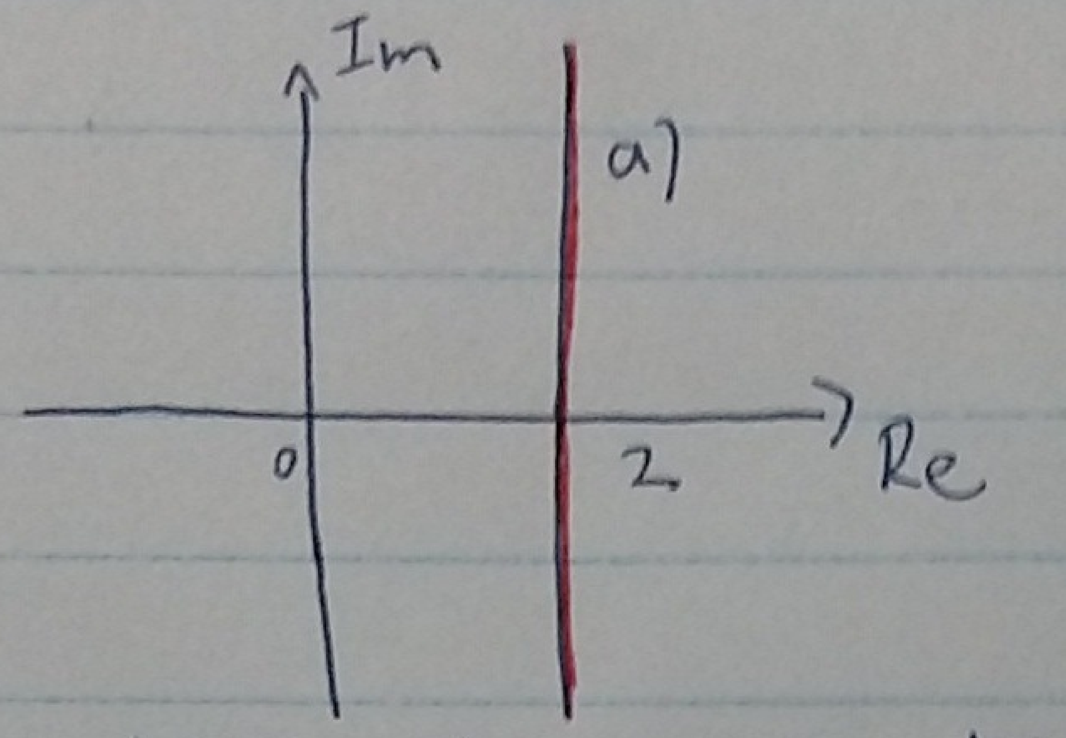
As a result, by M.I., the result is true $\forall n \in \mathbb{N}$.

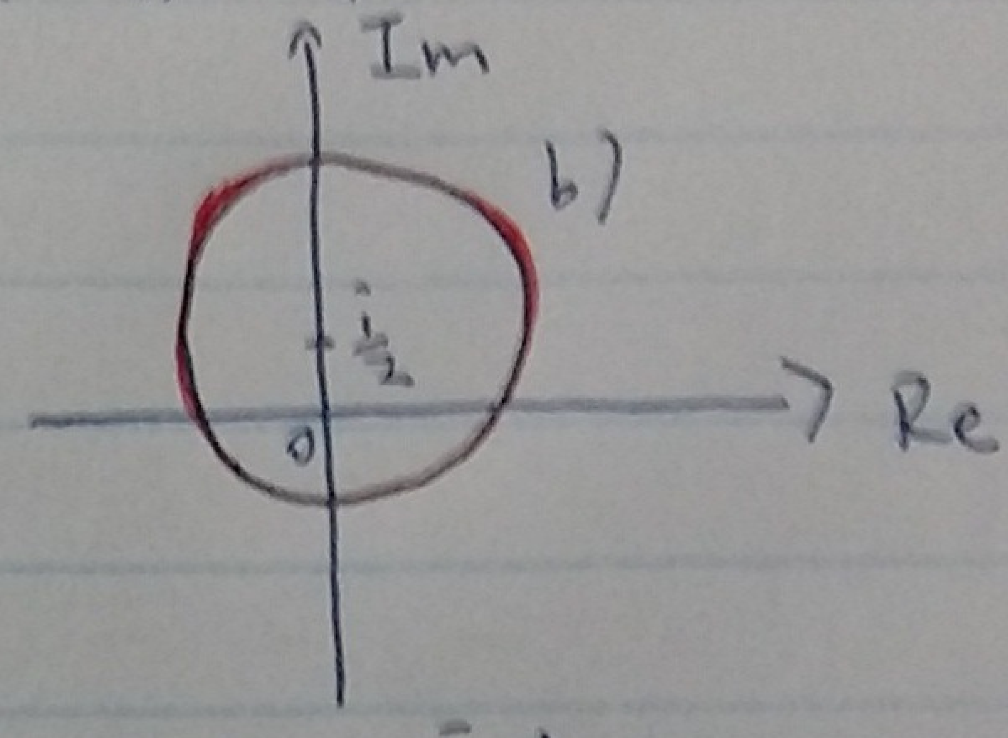
§4) 3) Note that (1) $|z_1|+|z_2| \geq Re(z_1)+Re(z_2) = Re(z_1+z_2)$
(2) $||z_3|-|z_4|| \leq |z_3+z_4|$,

we have $\dfrac{Re(z_1+z_2)}{|z_3+z_4|} \leq \dfrac{|z_1|+|z_2|}{||z_3|-|z_4||}$ for $|z_3| \neq |z_4|$.

§5) 2) a) Write $z = x+iy$.

Then $\quad Re(\bar{z}-i) = 2 \iff Re(x-(y+1)i) = 2 \iff x = 2.$





b) $\quad |2\bar{z}+i| = 4 \iff |\bar{z}+\frac{i}{2}| = 2 \iff |z-\frac{i}{2}| = 2$

13) $\quad |z-z_0| = R \iff |z-z_0|^2 = R^2$

$\iff (z-z_0)(\bar{z}-\bar{z}_0) = R^2$

$\iff z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = R^2$

$\iff |z|^2 - (z\bar{z}_0 + \overline{z\bar{z}_0}) + z_0\bar{z}_0 = R^2$

$\iff |z|^2 - 2Re(z\bar{z}_0) + |z_0|^2 = R^2$

§8) 2) a) $\quad |e^{i\theta}| = |\cos\theta + i\sin\theta| = \cos^2\theta + \sin^2\theta = 1$

b) $\quad \overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

5) a) $\quad i(1-\sqrt{3}i)(\sqrt{3}+i)$

$= 4i\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 4e^{i(\frac{\pi}{2})} e^{i(\frac{-\pi}{3})} e^{i(\frac{\pi}{6})}$

$= 4e^{i(\frac{\pi}{3})}$

$= 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= 2(1+\sqrt{3}i)$

b) Since $|2+i| = \sqrt{2^2+1^2} = \sqrt{5}$, let $(2+i) = \sqrt{5}e^{i\theta}$ for some $\theta \in (-\pi, \pi]$

Then $\dfrac{5i}{2+i} = \dfrac{5e^{i(\frac{\pi}{2})}}{\sqrt{5}e^{i\theta}} = \sqrt{5}e^{i(\frac{\pi}{2}-\theta)} = \sqrt{5}(\sin\theta + i\cos\theta) = 1+2i$

9) $\because$ $(1-z)(1+z+z^2+\cdots+z^n)$

$= 1+z+z^2+\cdots+z^n - z - z^2 - z^3 - \cdots - z^{n+1}$

$= 1 - z^{n+1}$

$\therefore$ $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$ for $z \neq 1$.

Put $z = e^{i\theta}$, we have

$$1 + e^{i\theta} + e^{i(2\theta)} + \cdots + e^{i(n\theta)} = \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} \quad \text{—— } (*)$$

Note that the real part of L.H.S. of $(*)$ is given by

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta.$$

For R.H.S., note that

$$\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} = \frac{e^{i(n+1)\theta}-1}{e^{i\theta}-1}$$

$$= \frac{e^{\frac{i(n+1)\theta}{2}} - e^{-\frac{i(n+1)\theta}{2}}}{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}} \times \frac{e^{\frac{i(n+1)\theta}{2}}}{e^{\frac{i\theta}{2}}}$$

$$= \frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} \cdot e^{i\left(\frac{n\theta}{2}\right)}$$

$\therefore$ Real part of R.H.S. of $(*)$ is given by

$$Re\left(\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}} \cdot e^{\frac{in\theta}{2}}\right) = \frac{\sin\left(\frac{n+1}{2}\right)\theta \cos\left(\frac{n}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta}{2} + \sin\frac{(2n+1)\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{1}{2} + \frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\frac{\theta}{2}}$$

By comparing the real part of both sides of $(*)$, we have

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin\left(\frac{2n+1}{2}\theta\right)}{2\sin\frac{\theta}{2}}, \quad \theta \in (0, 2\pi).$$

10) $\because$ $e^{i(3\theta)} = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$\therefore$ a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin\theta$

b) $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$.

10) 1) a) $\because 2i = 2e^{i(\frac{\pi}{2})}$

$\therefore (2i)^{\frac{1}{2}} = \sqrt{2}\, e^{i(\frac{\pi}{4})}$ or $\sqrt{2}\, e^{i(\frac{\pi}{4}+\pi)}$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$ or $\sqrt{2}\left(\frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)$

$= 1 + i$ or $-1 - i$,,

b) $\because (1-\sqrt{3}i) = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 2e^{i(-\frac{\pi}{3})}$

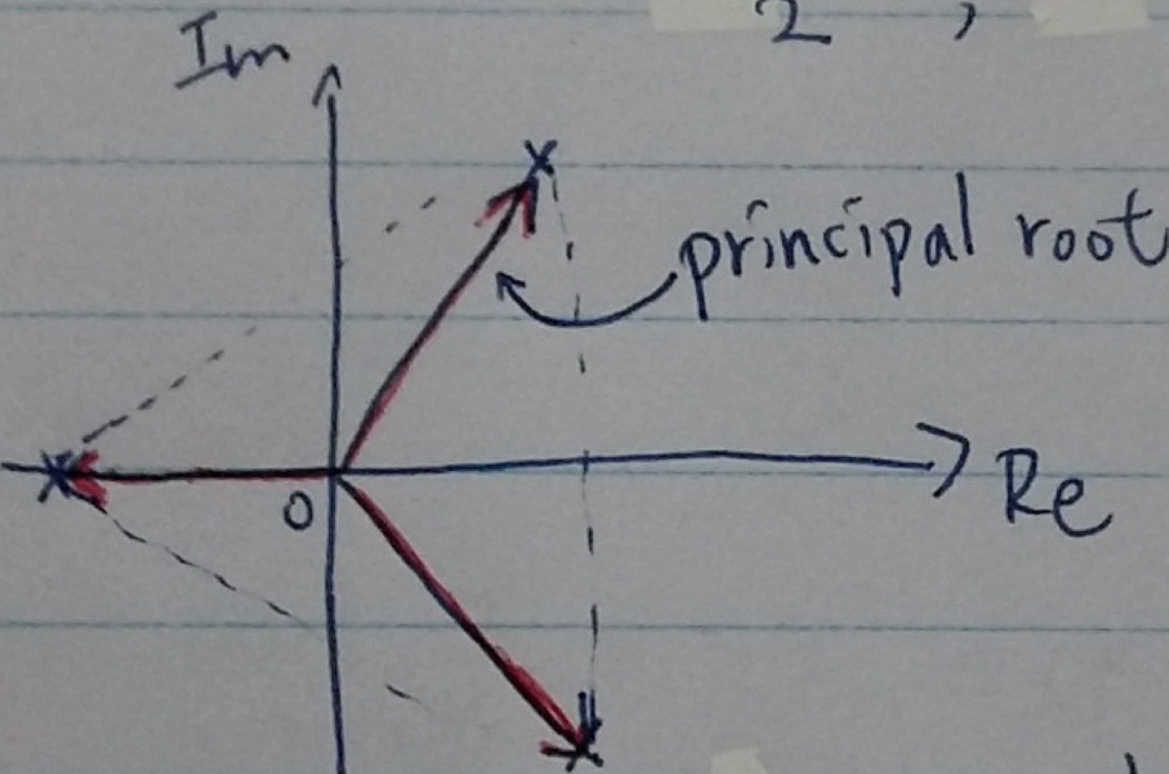
$\therefore (1-\sqrt{3}i)^{\frac{1}{2}} = \sqrt{2}\, e^{-i(\frac{\pi}{6})}$ or $\sqrt{2}\, e^{i(-\frac{\pi}{6}+\pi)}$

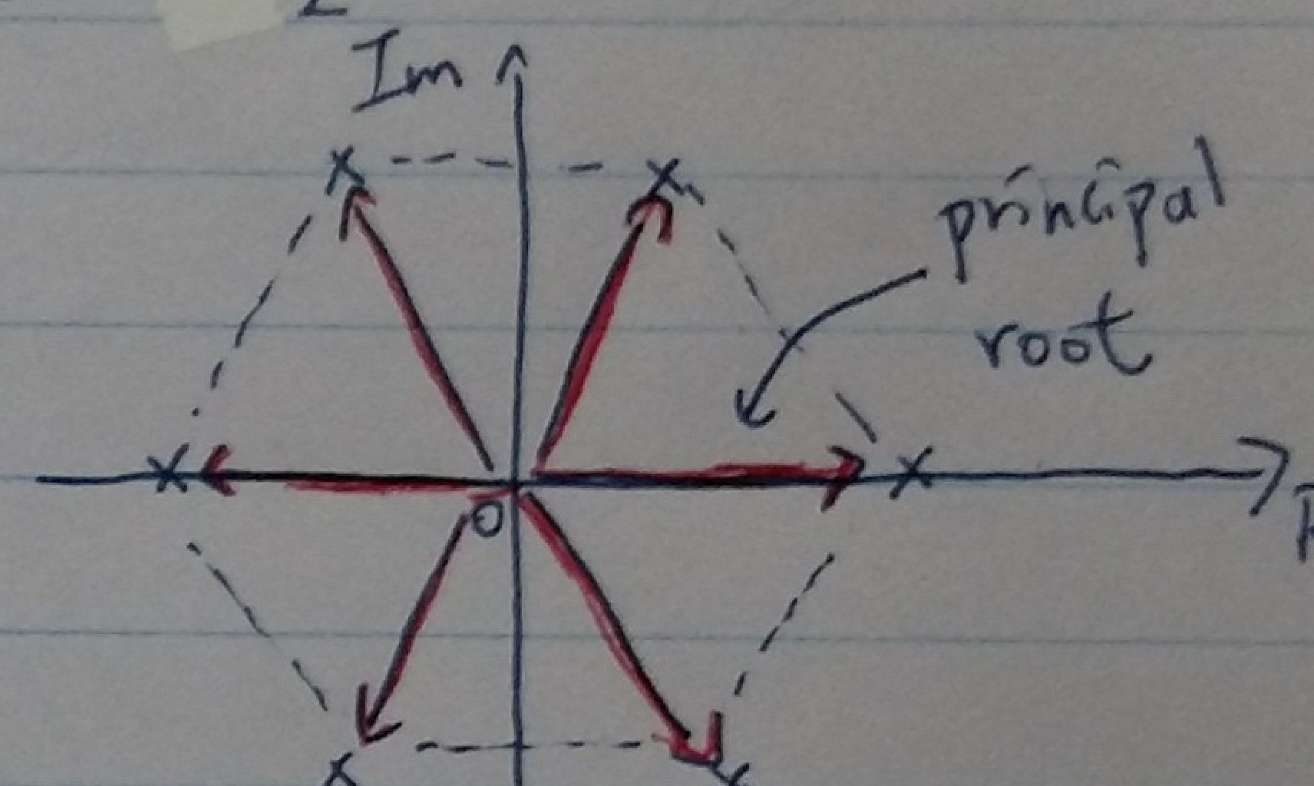
$= \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2} i\right)$ or $\sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)$

$= \frac{\sqrt{3} - i}{\sqrt{2}}$ or $-\left(\frac{\sqrt{3} - i}{\sqrt{2}}\right)$,,

3) a) $(-1)^{\frac{1}{3}} = (e^{i\pi})^{\frac{1}{3}}$

$= \frac{1+\sqrt{3}i}{2}$, -1 or $\frac{1-\sqrt{3}i}{2}$





b) $8^{\frac{1}{6}} = (8e^{i(0)})^{\frac{1}{6}}$

$= 8^{\frac{1}{6}}e^{i(0)}, 8^{\frac{1}{6}}e^{i(0+\frac{2\pi}{6})}, 8^{\frac{1}{6}}e^{i(0+\frac{4\pi}{6})}, 8^{\frac{1}{6}}e^{i(0+\frac{6\pi}{6})},$
$8^{\frac{1}{6}}e^{i(0+\frac{8\pi}{6})}, 8^{\frac{1}{6}}e^{i(0+\frac{10\pi}{6})}$

$= \sqrt{2}, \sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right), \sqrt{2}\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right), -\sqrt{2},$
$\sqrt{2}\left(\frac{-1}{2} - \frac{\sqrt{3}}{2} i\right), \sqrt{2}\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)$

$= \pm\sqrt{2}, \pm\frac{1+\sqrt{3}i}{\sqrt{2}}, \pm\frac{1-\sqrt{3}i}{\sqrt{2}}$

4) $z_0 = -4\sqrt{2} + 4\sqrt{2}i = 8\left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right) = 8e^{i(\frac{3\pi}{4})}$

$\therefore$ The principal cube root $c_0 = 8^{\frac{1}{3}} e^{i\frac{\pi}{4}} = 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

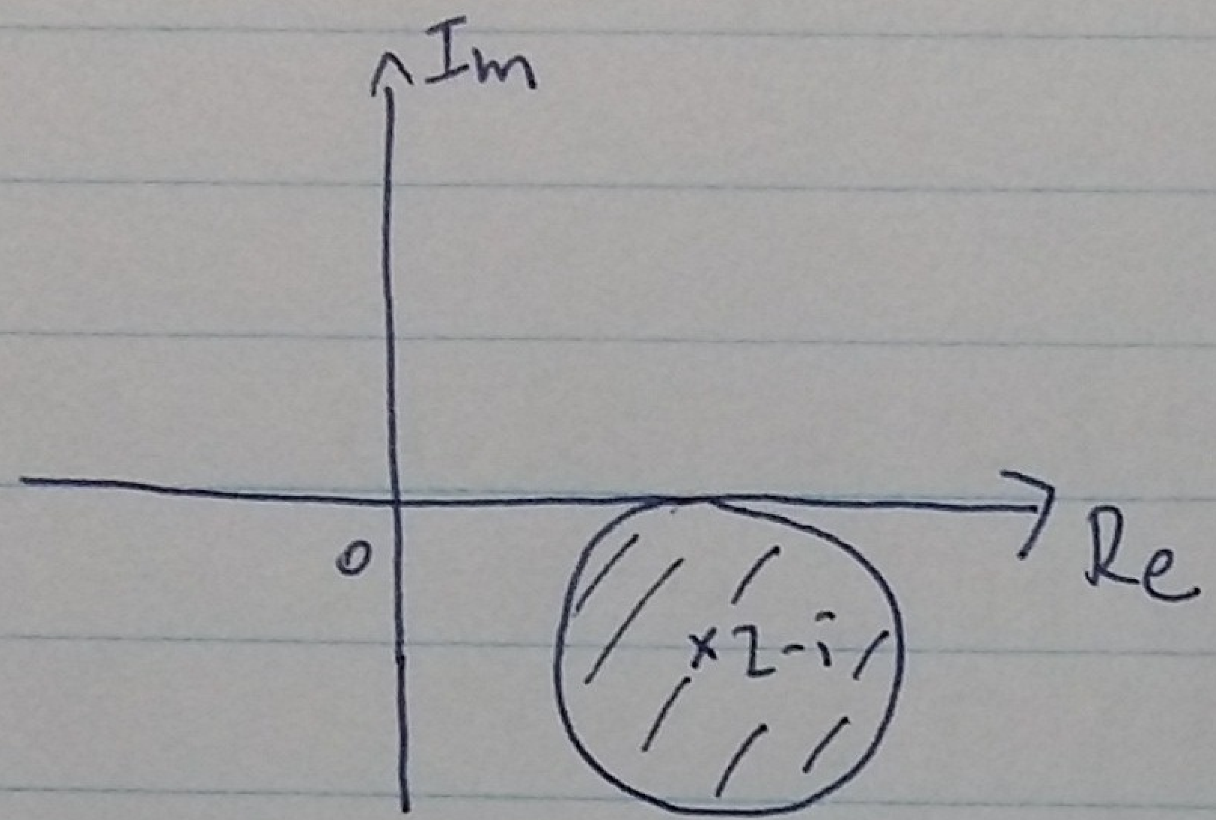
$\Rightarrow c_0 = \sqrt{2}(1+i)$

The other two cube roots are given by

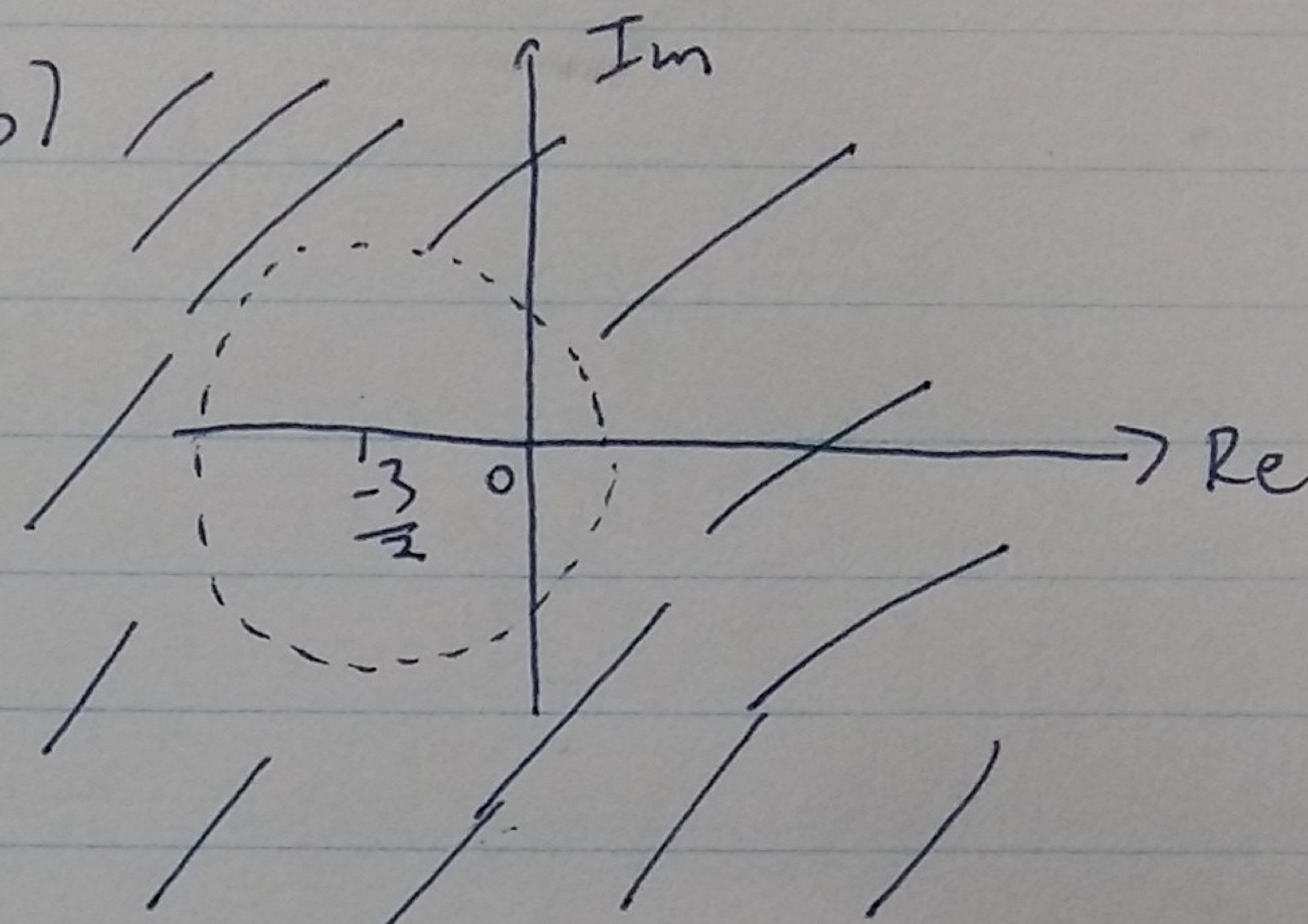
$c_0 w_3 = \sqrt{2}(1+i)\left(\frac{-1+\sqrt{3}i}{2}\right) = \frac{-(\sqrt{3}+1) + (\sqrt{3}-1)i}{\sqrt{2}}$

and $6\omega_3^2 = \sqrt{2}(1+i)\, e^{i(\frac{4\pi}{3})} = \sqrt{2}(1+i)\left(\frac{-1-\sqrt{3}i}{2}\right) = \frac{(\sqrt{3}-1)-(\sqrt{3}+1)i}{\sqrt{2}}$

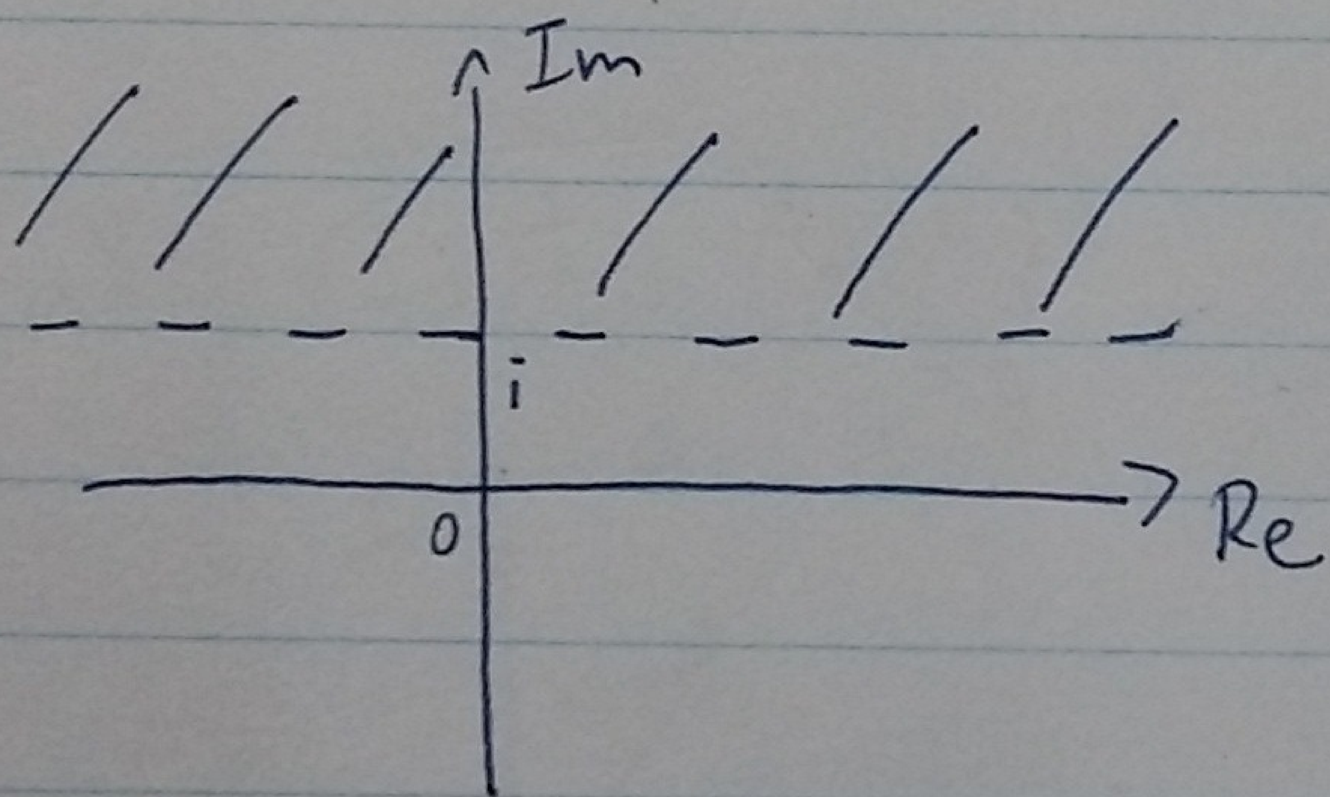
1) a)



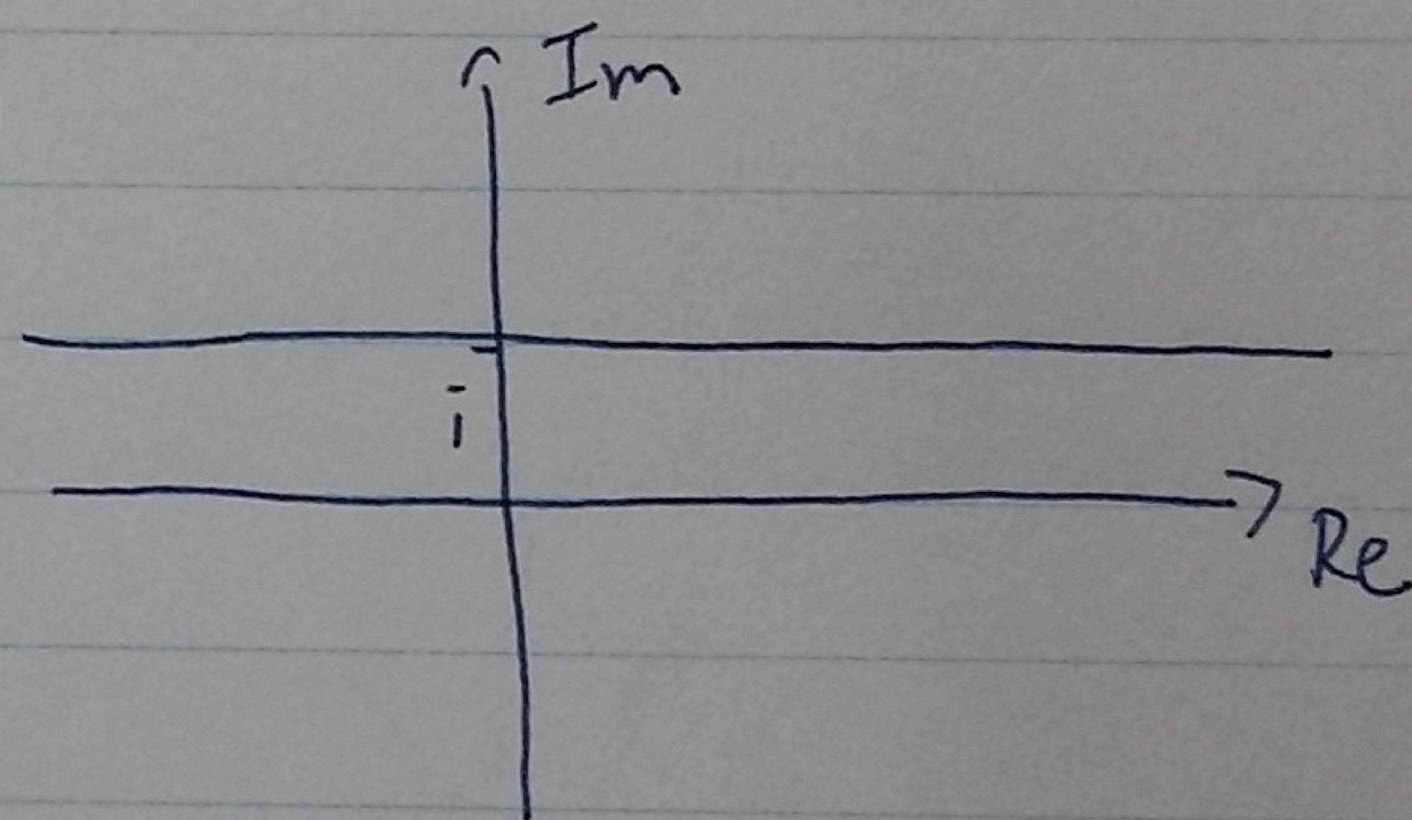
b)



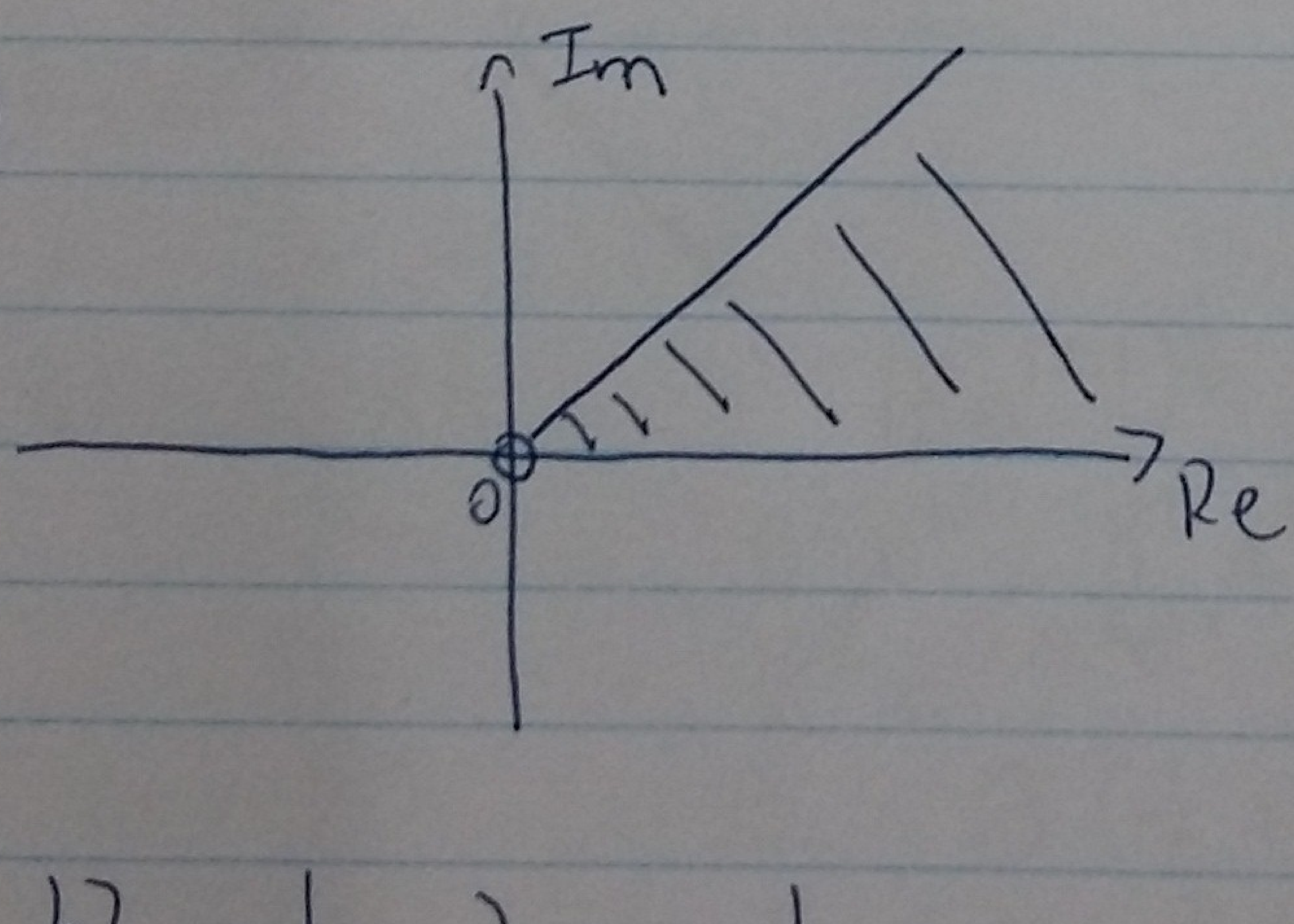
c)



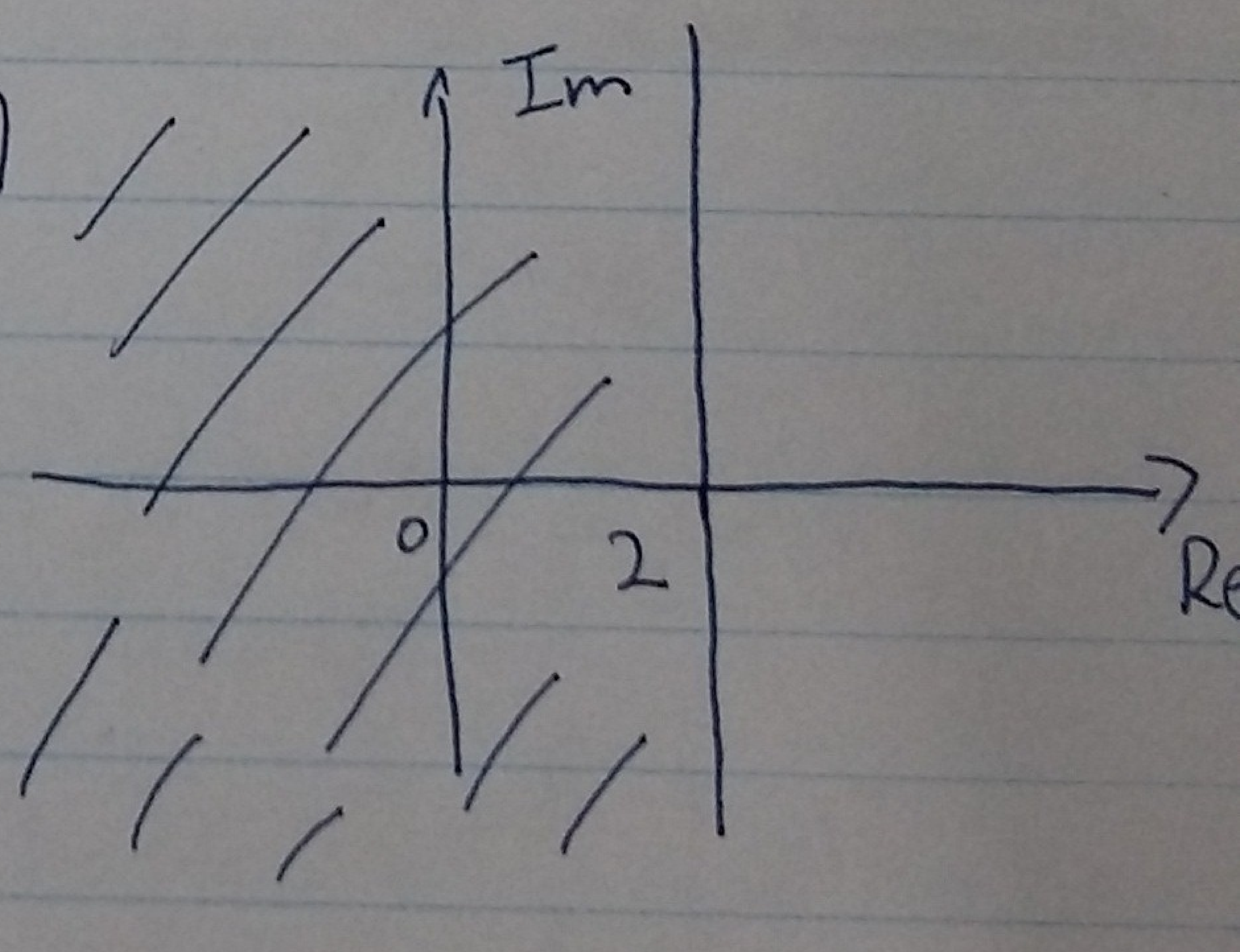
d)



e)



f)



b) and c) are domains.

2) e) is not open because it contains the points with arg $z = 0$ and arg $z = \frac{\pi}{4}$.
e) is not closed because it does not contain $(0,0)$, which is an accumulation point.

3) a) is clearly bounded.